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Abstract

We consider a finite difference approximation to the Cauchy problem for a first-order hyperbolic partial differential equation using different mesh spacings in different portions of the domain. By reformulating our problem as a difference approximation to an initial-boundary value problem, we are able to use the theory of H. O. Kreiss and S. Osher to analyze the L_2 stability of our scheme.

In the case of one space dimension, a proof is presented for the L_2 stability of the Lax-Wendroff scheme when the interface condition is generated so as to give second order accuracy. Stability is also proven for all 3-point dissipative schemes which are matched along an interface by various conditions.

For the case of two space dimensions, stability is proven with an analogous mesh refinement performed along a coordinate line. Along this line data is generated by interpolation.

Numerical experiments were performed on the CDC-6600. We computed test problems for the schemes discussed in our theoretical treatment, but also introduced an additional feature. We used uneven time steps in the different mesh patterns. Our numerical results indicate that our techniques are an efficient method for improving accuracy in regions of special interest, while preventing new inaccuracies at the interface.

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Acknowledgements

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INTRODUCTION.

There are many situations in the numerical solution of partial differential equations by finite difference approximations where one wishes to perform the calculations with different mesh spacings in different portions of the domain. This paper considers the L_2 stability of a finite difference solution to the Cauchy problem

$$
(0.1) \qquad \begin{aligned} u_t &= A\, u_x + B\, u_y \;, & t &\geq 0 \\ u(x,y,0) &= \Phi(x,y) \;, & -\infty &< x,y < \infty \end{aligned}
$$

where the spatial mesh sizes have been refined in a portion of the domain. Problems of this type shall be referred to as mesh refinement problems in this paper.

Chapter I contains a review of the terminology and basic theory of stability for pure initial-value and initial-boundary value problems. A specific mesh refinement problem is formulated for the one dimensional case of (0.1). Stability is analyzed by employing the notion of local normal modes as described by Richtmyer and Morton [11,12] in their exposition of the work of Godunov-Ryabenkii [1,2]. The technically crucial result in Lemma 1 states that the modes of dissipative equations are separated by the unit circle.

In Chapter II the problem is reformulated as a

difference approximation to an initial-boundary value problem for a system of partial differential equations. The boundary condition employed is derived from the particular approximation used to define the difference equation at the point of contact between the two grid patterns. General sufficient conditions for the stability of such systems are due to H. O. Kreiss [6] and S. Osher [9,10]. In Chapter II, we review those aspects of the Kreiss-Osher results which we use in this paper.

In Chapter II, for the case of one space variable, we show the L_2 stability of a generalization of the Lax-Wendroff scheme to the mesh refinement technique. In Theorem IV, we give a proof of the stability of all 3 point dissipative schemes to the one dimensional case of (0.1) using various interface conditions. This result allows us to match two schemes in a stable way along an interface.

The case of two space variables is treated in Chapter III. Using the results of Kreiss and Osher, we are able to prove stability when the mesh refinement is performed along a line parallel to an axis, when our interface condition is generated by interpolating the data along this line.

Numerical results obtained on the CDC-6600 are presented in Chapter IV. We use these results to compare the accuracy of the various approximations discussed.

The numerical results bear out our theoretical conclusions, and, furthermore yield evidence of stability for cases not analytically treated (e.g. variable coefficients, refinements having corners, and refinements using different time steps in different portions of the domain).

CHAPTER I. STABILITY THEORY.

a. Initial-value problems.

We consider the numerical solution of the initial-value problem

$$(1.1) \qquad \begin{aligned} & u_t = A\, u_x \quad , \qquad -\infty < x < \infty, \quad t \geq 0 \,; \\ & u(x,0) = \Phi(x) \quad , \end{aligned}$$

where A is a symmetric matrix, and $\Phi(x)$ is the prescribed initial value function of the vector valued solution $u(x,t)$. Let $\Phi(x)$ be sufficiently smooth so that (1.1) possesses a solution operator which is bounded in L_2 for all finite t. In such a case, we say that the problem (1.1) is properly posed.

We introduce a set of uniform mesh points

$$(1.2) \qquad R = \left\{ (x,t) : \begin{array}{lll} x = jh & \text{for} & j = 0, \pm 1, \pm 2, \ldots \\ t = nk & \text{for} & n = 0,\ 1,\ 2, \ldots \end{array} \right\}$$

where $h = \Delta x$ is the spatial mesh size and $k = \Delta t$ denotes the time step. Throughout this paper we assume

$$k/h = \text{constant.}$$

Consider an explicit finite difference approximation to (1.1) of the form

$$v(x,t+k) = S_h\, v(x,t) = \sum_j s_j\, v(x+jh,t)$$
(1.3)
$$v(x,0) = u(x,0)$$

where the s_j are constants, and $(x,t) \in R$.

When reference is made to a norm in this paper, we shall always mean the L_2 norm,

(1.4) $$\|u(t)\| = \sqrt{\sum_{(x,t)\in R} |u(x,t)|^2} \quad ; \quad t \text{ fixed.}$$

Definition. The difference scheme (1.3) approximates (1.1) with m-th order accuracy if for all sufficiently smooth solutions $u(x,t)$ of (1.1)

(1.5) $$\|u(x,t+k) - S_h u(x,t)\| = O(h^{m+1}) .$$

S_h is said to be consistent if $m \geq 1$.

Definition. The difference scheme (1.2) is stable if

(1.6) $$\|S_h^n\| \leq K ,$$

K some constant for all n,k satisfying $n \cdot k \leq 1$.

The following theorem is well known [12]: Convergence of a consistent difference scheme to the solution of a properly posed initial-value problem is equivalent to the stability of the difference scheme.

Schemes with accuracy of order m can be constructed

with the aid of Taylor's theorem. For $u(x,t)$ a solution to (1.1) where A is independent of t, we can express the time derivatives as spatial derivatives in the Taylor expansion, so that

$$(1.7) \quad u(x,t+k) = \sum_{\ell=0}^{m} \frac{k^{\ell}}{\ell!} \left(A \frac{\partial}{\partial x}\right)^{\ell} u(x,t) + O(k^{m+1}) .$$

In (1.3) we expand $v(x+jh,t)$ in a Taylor series about x. Equating terms of order m or less in (1.3) and (1.7) yields a system of m +1 equations (powers of h) for the m +1 unknowns, $s_0, s_1, \dots, s_m$, the coefficients of S_h in (1.3).

An alternate approach is to verify (1.5) for a dense set of smooth data. One of the advantages of this method is that one is led to consider the following function related to (1.3)

$$(1.8) \qquad C(\xi) = \sum_j s_j \, e^{ij\xi}$$

called the symbol (amplification matrix) of the operator S_h. As shown in [7], we can study stability of S_h explicitly in terms of the uniform boundedness of the matrices $C^n(\xi)$.

A well known necessary condition for stability is the von-Neumann condition: <u>the eigenvalues of</u> C <u>do not exceed one in absolute value</u>.

In the future we shall refer to the following

initial-value problem as:

Example I.

$$(1.9)\qquad \begin{aligned} u_t &= a\,u_x\,, \qquad -\infty < x < \infty \\ u(x,0) &= \Phi(x) \end{aligned}$$

where a is a constant and u and Φ are scalar functions. One can verify that the consistency conditions for (1.3) give for $b = a\,\Delta t/\Delta x$

$$\sum_j j^{\ell} s_{\ell} = b^{\ell}\,, \qquad \ell = 0,1,\ldots,m.$$

If we use a symmetrically centered scheme of $2N+1$ points, we are led to take $m = 2N$. For the case $N = 1$, the Lax-Wendroff scheme gives us second order accuracy using as few points as possible. For future reference we note that for this scheme

$$(1.10)\qquad s_{-1} = \frac{b^2-b}{2}\,, \qquad s_0 = 1 - b^2\,, \qquad s_1 = \frac{b^2+b}{2}$$

where $b = a\Delta t/\Delta x$ is a constant.
Using (1.10) we find as in [8] the following equation for the absolute value of the amplification factor squared,

$$(1.11)\qquad |C(\xi)|^2 = 1 - (b^2-b^4)(1-\cos\,\xi)^2\,.$$

Thus the von-Neumann condition (which is also a sufficient condition for stability in the scalar case [12]) is satisfied iff $|b| \leq 1$.

b. <u>A simple mesh refinement problem</u>.

Consider the Cauchy problem of Example I. Suppose that we attempt a numerical solution using two different patterns of mesh spacings. On the right hand side of the origin the mesh length is Δx_1; on the left hand side, we take the spatial mesh size to be Δx_2. Setting

$$p = \frac{\Delta x_1}{\Delta x_2} \tag{1.12}$$

we assume that, say, the mesh on the right hand side is more *refined* than on the left, i.e.

$$0 < p < 1 .$$

Set

$$b_1 = \frac{a\,\Delta t}{\Delta x_1} \quad \text{and} \quad b_2 = \frac{a\,\Delta t}{\Delta x_2} \tag{1.13}$$

so that

$$b_2 = p\, b_1 .$$

Let w_j^n denote our difference approximation to the solution $u(x,t)$ of Example I at $x = j\Delta x_1$, $t = n\Delta t$ for $j = 0,1,2,\ldots$; $n = 0,1,2,\ldots$. Then for $j = 1,2,3,\ldots$,

$$w_j^{n+1} = \left(\frac{b_1^2-b_1}{2}\right)w_{j-1}^n + (1-b_1^2)w_j^n + \left(\frac{b_1^2+b_1}{2}\right)w_{j+1}^n . \tag{1.14a}$$

The above formula also applies as an approximation to $u(j\,\Delta x_2,\, n\,\Delta t)$ for j negative if we replace b_1 by b_2. But, regardless of the approximation chosen for w_0^{n+1}, the

difference equations have a discontinuity at $x = 0$ in the sense that they approximate the differential equation (1.9) where the coefficient $a = a(x)$ is a piecewise constant function with a jump at $x = 0$. We avoid discussing this by reducing our problem to an initial-boundary value problem in the quarter plane. To this end, we introduce a new difference function v_j^n which denotes an approximation to $u(-j\,\Delta x_2,\ n\,\Delta t)$ for $j \geq 0$. Then for $j = 1,2,3,\ldots,$ and $n = 0,1,2,\ldots,$

$$(1.14)\text{b} \quad v_j^{n+1} = \left(\frac{b_2^2+b_2}{2}\right)v_{j-1}^n + (1-b_2^2)v_j^n + \left(\frac{b_2^2-b_2}{2}\right)v_{j+1}^n \,.$$

We introduce the vector notation

$$(1.15) \qquad Z_j^n = \begin{pmatrix} w_j^n \\ v_j^n \end{pmatrix} \;; \quad j = 0,1,2,\ldots;\ n = 0,1,2,\ldots,$$

and note that by our definitions

$$v_0^n \equiv w_0^n \,.$$

Then (1.14)a,b can be expressed as

$$(1.16) \qquad Z_j^{n+1} = Q\, Z_j^n$$

where

$$
(1.17)\quad Q = \begin{pmatrix} \frac{b_1^2-b_1}{2} & 0 \\ 0 & \frac{p^2b_1^2+pb_1}{2} \end{pmatrix} T^{-1} + \begin{pmatrix} 1-b_1^2 & 0 \\ 0 & 1-p^2b_1^2 \end{pmatrix} T^0 + \begin{pmatrix} \frac{b_1^2+b_1}{2} & 0 \\ 0 & \frac{p^2b_1^2-pb_1}{2} \end{pmatrix} T^1 ,
$$

where we have used (1.13) and where the T^k, the translation operators, are such that $T^{\ell}Z_j^n = Z_{j+\ell}^n$.

There remains to prescribe an approximation to $u(0,t)$ at each time step. In general this would take the form

$$
(1.18)\quad w_0^{n+1} = \sum_{j=0}^{s} \alpha_j \, w_j^n + \sum_{j=1}^{t} \beta_j \, v_j^n \; ; \qquad s,t \quad \text{integers.}
$$

We consider the case $0 < p < 1$, where we make the natural requirement that our difference scheme be of second order accuracy at the point $x = 0$. Using the consistency conditions derived from Taylor's theorem, we solve for $\alpha_0, \alpha_1, \beta_1$ in

$$
(1.19)\qquad w_0^{n+1} = \alpha_0 w_0^n + \alpha_1 w_1^n + \beta_1 v_1^n + O(\Delta x_2^3)
$$

an unevenly spaced scheme. A straightforward calculation yields the following set of equations;

$$
(1.20) \qquad \begin{aligned} \alpha_0 + \alpha_1 \;+ \beta_1 &= 1 \\ \alpha_1 p - \beta_1 &= b_2 \\ \alpha_1 p^2 + \beta_1 &= b_2^2 \end{aligned} \quad .
$$

Solving and substituting into (1.19) we have up to second order accuracy

$$
(1.21) \qquad \begin{aligned} w_0^{n+1} = {} & \frac{b_2(b_2 - p)}{(1+p)} v_1^n + \left(1 - \frac{b_2}{p}\right)(1 + b_2) w_0^n \\ & + \frac{b_2(b_2+1)}{p(p+1)} w_1^n \; ; \quad n = 0,1,2,\ldots \; . \end{aligned}
$$

We can express (1.21) in the following vector form

$$
(1.22) \qquad z_0^{n+1} = \begin{pmatrix} \alpha_0 & 0 \\ \alpha_0 & 0 \end{pmatrix} z_0^n + \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_1 & \beta_1 \end{pmatrix} z_1^n \; .
$$

absolute value then we expect that one could construct an unstable mode ($|\lambda| > 1$) by using

$$(1.26) \qquad \underline{\Phi}_j = \begin{pmatrix} c_1\mu_1^j + c_2\mu_2^j \\ d\nu_1^j \end{pmatrix}$$

which could be chosen to satisfy the boundary conditions (1.22). That this cannot be the case is due to the fact that the μ_1, μ_2 (ν_1, ν_2) are separated by the unit circle so that always, say, $|\mu_1| < 1$ and $|\mu_2| > 1$ for $|\lambda| > 1$.

d. Separation of the modes.

Consider a consistent three point difference approximation to (1.9) for $n = 0,1,2,\ldots;\ j = 0,\pm1,\pm2,\ldots$

$$(1.27) \qquad w_j^{n+1} = a_{-1}w_{j-1}^n + a_0 w_j^n + a_1 w_{j+1}^n$$

where the a_i are constants. Let $b = a\,\frac{\Delta t}{\Delta x}$ be a constant. We check the von-Neumann condition by substituting into (1.27) a solution of the form

$$(1.28) \qquad w_j^n = c\ \mu^j \lambda^n, \qquad c,\ \text{constant},$$

where $\mu = e^{i\xi}$, ξ real, $|\xi| \leq \pi$.

We get a quadratic equation in μ for the modes;

$$(1.29) \qquad a_1\mu^2 + (a_0 - \lambda)\mu + a_{-1} = 0 .$$

The difference scheme (1.27) is stable iff $|\mu| = 1$ implies $|\lambda| \leq 1$.

Lemma 1 (Separation of roots).

Let the difference scheme (1.27) be stable, so that

$$(1.30) \quad |a_{-1}\, e^{-i\xi} + a_0 + a_1\, e^{i\xi}| \leq 1, \quad |\xi| \leq \pi, \; \xi \text{ real.}$$

Then the μ roots of (1.29) are separated for any λ, $|\lambda| > 1$, which means that one of the roots of (1.29) lies inside the unit circle, and the other outside. (Note, more general results of this form can be found in [6] and [9].)

Proof. Let μ_1, μ_2 be the roots of (1.29), then

$$\mu_1 \mu_2 = a_{-1}/a_1 \qquad \text{and} \qquad \mu_1 + \mu_2 = \frac{a_0 - \lambda}{-a_1} .$$

Taking the limit as λ tends to infinity, we observe that the product remains constant, but the sum blows up. Thus, we conclude that one root tends to infinity and the other to zero. That is, the unit circle separates the roots as $\lambda \to \infty$. That this is true for any λ_0 such that $|\lambda_0| > 1$ follows from the fact that the roots $\mu_1(\lambda)$, $i = 1,2$, depend continuously on λ. Indeed, as λ tends from infinity to λ_0 through values of absolute value greater than one, the values $\mu_1(\lambda)$, $\mu_2(\lambda)$ cannot touch the unit circle, for then the stability criterion would imply that

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We get a quadratic equation in μ for the modes;

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Then the μ roots of (1.29) are separated for any λ, $|\lambda| > 1$, which means that one of the roots of (1.29) lies inside the unit circle, and the other outside. (Note, more general results of this form can be found in [6] and [9].)

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$$\mu_1\mu_2 = a_{-1}/a_1 \qquad \text{and} \qquad \mu_1 + \mu_2 = \frac{a_0 - \lambda}{-a_1}\,.$$

Taking the limit as λ tends to infinity, we observe that the product remains constant, but the sum blows up. Thus, we conclude that one root tends to infinity and the other to zero. That is, the unit circle separates the roots as $\lambda \to \infty$. That this is true for any λ_0 such that $|\lambda_0| > 1$ follows from the fact that the roots $\mu_i(\lambda)$, $i = 1,2$, depend continuously on λ. Indeed, as λ tends from infinity to λ_0 through values of absolute value greater than one, the values $\mu_1(\lambda)$, $\mu_2(\lambda)$ cannot touch the unit circle, for then the stability criterion would imply that

$$|\lambda| \leq 1 .$$

Thus μ_1 and μ_2 remain separated by the unit circle for all λ, $|\lambda| > 1$. Q.E.D.

By consistency, $a_{-1} + a_0 + a_1 = 1$. So $\mu = 1$ implies $\lambda = 1$. However, the following is easily proven by a similar argument as in Lemma 1.

<u>Corollary</u> 1. If for $0 < |\xi| \leq \pi$ equation (1.30) is satisfied with strict inequality then the unit circle separates the two μ roots of (1.29) for $|\lambda| \geq 1$, $\lambda \neq 1$.

We see from (1.11) that the Lax-Wendroff scheme satisfies the hypothesis of the above corollary, and thus all L_2 solutions of the form (1.23), (1.24) to (1.16), (1.17) must have the simple form

$$(1.24)' \qquad \underline{\Phi}_j = \begin{pmatrix} c\ \mu^j \\ d\ \nu^j \end{pmatrix}, \quad j = 0,1,2,\ldots;\ c,d,\ \text{const.}$$

A necessary condition for stability of our mesh refinement problem of Section b. is that (1.23), (1.24)' does not satisfy (1.22) for $|\lambda| > 1$.

It can be shown that the interface condition according to (1.22) rules out the possibility of having $|\lambda| > 1$ for our solution (1.24)'. No verification is made here, because this result will be a consequence of our proof of sufficiency conditions for stability in Theorem I.

CHAPTER II. MESH REFINEMENTS AS INITIAL-BOUNDARY VALUE PROBLEMS.

a. The theory of Kreiss and Osher.

In this section, we present the theory of Kreiss [6] and Osher [9,10] for initial-boundary value problems specialized to a simple case which suffices for our purposes. Consider a first-order system of partial differential equations

$$(2.1)\quad \frac{\partial u}{\partial t} = A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y}\ ,\ t \geq 0\ ,\ 0 \leq x < \infty\ ,\ -\infty < y < \infty,$$

where A and B are constant diagonal matrices. A has the form

$$(2.2)\quad A = \begin{pmatrix} a_1 & & & & & & \bigcirc \\ & a_2 & & & & & \\ & & \ddots & & & & \\ & & & a_K & & & \\ & & & & a_{K+1} & & \\ & & & & & \ddots & \\ \bigcirc & & & & & & a_M \end{pmatrix} \qquad \begin{matrix} a_1 < 0,\ldots,a_K < 0 \\ a_{K+1} > 0,\ldots,a_M > 0. \end{matrix}$$

We consider an initial-boundary value problem in the quarter space $x \geq 0$, $t \geq 0$.

$$(2.3)\quad u(x,y,0) = \phi(x,y)\ ,\quad 0 \leq x < \infty\ ,\quad -\infty < y < \infty\ .$$

$$(2.4)\text{a)}\quad u^{I}(0,y,t) = S_{II}\, u^{II}(0,y,t)\ ,$$

where S_{II} is a constant rectangular matrix, and where

$$u^{I} = (u^{(1)}(x,y,t),\ldots,u^{(K)}(x,y,t))$$

(2.4) b)

$$u^{II} = (u^{(K+1)}(x,y,t),\ldots,u^{(M)}(x,y,t)).$$

We approximate (2.1)-(2.4) by the system of difference equations

$$\text{(2.5) a)} \qquad v^{n+1}_{\ell,j} = \sum_{\alpha=-s}^{t_1} \sum_{\beta=-q}^{t_2} C_{\alpha,\beta}\, v^{n}_{\ell+\alpha,j+\beta}$$

for $n = 0,1,2,\ldots$; $\ell = s,s+1,\ldots$; $j = \ldots -1,0,1,\ldots$ with boundary conditions

$$\text{(2.5) b)} \qquad v^{n}_{\ell,j} = \sum_{\alpha=s}^{r+s} \sum_{\beta=-r_1}^{r_2} b_{\ell,\alpha,\beta}\, v^{n}_{\ell+\alpha,j+\beta}$$

for $\ell = 0,-1,\ldots,-s+1$; $j = \ldots -1,0,1,\ldots$; $n = 0,1,2,\ldots$ where q,r,r_1,r_2,s,t_1,t_2 are all non-negative integers, and

$$\text{(2.6)} \qquad v^{n}_{\ell,j} = (v^{n}_{\ell,j}(1),\ldots,v^{n}_{\ell,j}(M))$$

is an M component vector approximation to $u(\ell\Delta x, j\Delta y, n\Delta t)$. The $C_{\alpha,\beta}$ are constant diagonal $(M \times M)$ matrices and the $b_{\ell,\alpha,\beta}$ are constant $(M \times M)$ matrices.

We assume that a time step $k = \Delta t > 0$ and mesh sizes $h_1 = \Delta x > 0$ and $h_2 = \Delta y > 0$ have been introduced, so that $b_1 = k/h_1$ and $b_2 = k/h_2$ are constants.

The following three conditions are natural assumptions for the stability of (2.5):

1) The difference approximation (2.5) with its boundary conditions is consistent with the initial-boundary value problem for the differential equation (2.1).

2) The difference approximation defined by (2.5)a is L_2 stable in the pure Cauchy case.

3) The difference operator $C = \sum_{\alpha,\beta} C_{\alpha,\beta} T_x^{\ell+\alpha} T_y^{j+\beta}$ defined by (2.5) is <u>dissipative</u> [5] in the following sense: the eigenvalues $\lambda = \lambda(\xi)$ of the amplification matrix $\hat{C} = \sum_{\alpha,\beta} C_{\alpha,\beta} e^{i((\ell+\alpha)\xi_1+(j+\beta)\xi_2)}$, the Fourier transform of C, are strictly less than one for ξ_1, ξ_2 real and $0 < \sqrt{\xi_1^2 + \xi_2^2} \leq \pi$.

A sufficient condition for stability is found in:

<u>Theorem</u> (Kreiss-Osher).

Suppose that the assumptions 1-3 are valid. Suppose further that (2.5)a,b have no non-trivial generalized solutions of the form

$$v_{\ell,j}^n = \lambda^n \underline{\Phi}_\ell y_0^j, \qquad |y_0| = 1, \tag{2.7}$$

for which either a) or b) or c) holds, where

(2.8)

a) $|\lambda| \geq 1$, $\lambda \neq 1$ and $y_0 \neq 1$ and $\sum_{\ell=0}^{\infty} |\underline{\Phi}_\ell|^2 < \infty$

b) $|\lambda| = 1$, $\lambda \neq 1$, $y_0 = 1$ and $\sum_{\ell=0}^{\infty} |\underline{\Phi}_\ell|^2 < \infty$

c) $\lambda = y_0 = 1$ and $|\underline{\Phi}_\ell^{I}| \leq$ constant,

$|\underline{\Phi}_\ell^{II}| \leq$ constant $|\mu|^\ell$ where $|\mu| < 1$.

Then the approximation (2.5) is stable. (See (2.4)b for the definition of $\underline{\overline{\Phi}}_{\ell}^{I}$, $\underline{\overline{\Phi}}_{\ell}^{II}$.)

Note that the above theorem holds for the one-dimensional case $B \equiv 0$ with the obvious simplification $y_0 \equiv 1$, [6,9].

b. Stability of a mesh refinement problem.

We return to the one-dimensional mesh refinement problem of Chapter I,b. Equations (1.14)-(1.17) remain with only one minor change. As before, w_j^n denotes an approximation to $u(j\,\Delta x_1, n\,\Delta t)$ for $j = 0,1,2,\ldots;$ $n = 0,1,2,\ldots,$ but now in order to conform with the formulation of Kreiss and Osher, we found it necessary to change our notation so that v_j^n denotes an approximation to $u(-(j-1)\Delta x_2, n\,\Delta t)$ for $j = 0,1,2,\ldots;$ $n = 0,1,2,\ldots$. (See figure below.)

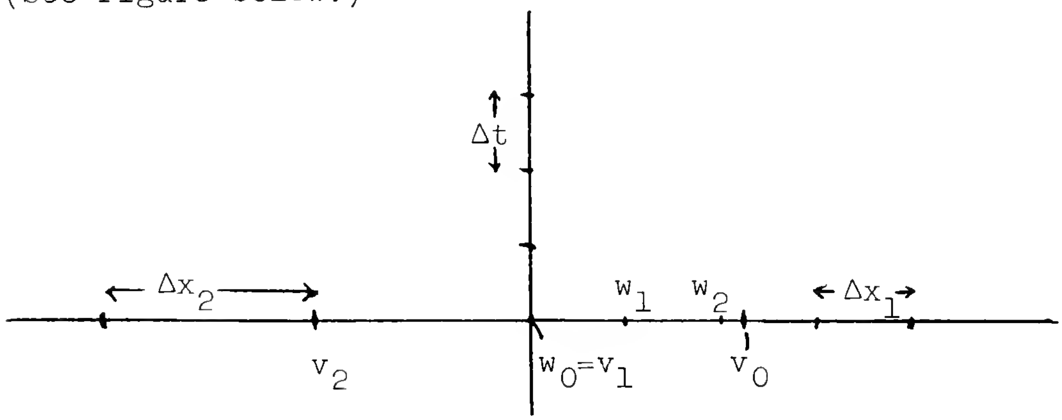

Figure 1

From our definitions of v_j^n and w_j^n we have

$$(2.9) \qquad w_0^n = v_1^n , \qquad n = 0,1,2,\ldots .$$

In general, a boundary condition of the type (2.5)b represents an extrapolation along a given time step. What is missing in our case is a formula for v_0^n. We obtain this by second-order extrapolation along the x-axis:

$$(2.10) \quad v_0^n = \alpha_1 w_1^n + \alpha_2 v_1^n + \alpha_3 v_2^n + O(\Delta x^3) .$$

Expanding w_1^n, v_1^n, v_2^n about the interface and using (2.10) to achieve second order extrapolation, we obtain the following equations for the α_i.

$$(2.11) \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 1 \\ (1-p)\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \quad \text{where } p = \dfrac{\Delta x_1}{\Delta x_2} \\ \dfrac{(1-p)^2}{2}\alpha_1 + \dfrac{1}{2}\alpha_2 + 2\alpha_3 = 0 \end{cases}$$

Solving (2.11) we get

$$(2.12) \quad \alpha_1 = \frac{2}{p(1+p)} , \quad \alpha_2 = \frac{2(p-1)}{p} , \quad \alpha_3 = \frac{1-p}{1+p} .$$

We take as a defining approximation for $n = 0,1,2,\ldots,$

$$(2.13) \quad v_0^n = \frac{2}{p(1+p)} w_1^n + \frac{2(p-1)}{p} v_1^n + \left(\frac{1-p}{1+p}\right) v_2^n .$$

Using the above equation for v_0^n in (1.16), we find that v_1^{n+1} is then determined as

$$(2.14)\qquad \begin{aligned} v_1^{n+1} &= \frac{(b_2^2 + b_2)}{p(1+p)} w_1^n + (1 - b_2/p)(1 + b_2)v_1^n \\ &\quad + \frac{b_2(b_2 - p)}{(1+p)} v_2^n , \qquad n = 0,1,2,\ldots . \end{aligned}$$

When one looks at the definitions for v_j^n and w_j^n in Chapter I,b and here, the above expression for v_1^{n+1} is seen to agree exactly with equation (1.21), which is a direct derivation of v_1^{n+1} as a linear combination of w_1^n, v_1^n, v_2^n retaining second-order accuracy at $x = 0$. Thus the formulation of this section is equivalent to that of Chapter I,b.

In order to apply Kreiss' theorem, we need only give a vector formulation of (2.9) and (2.13) which will serve as our boundary condition for (1.16), (1.17). Namely,

$$(2.15)\quad Z_0^n = \begin{pmatrix} w_0^n \\ v_0^n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} w_1^n \\ v_1^n \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \alpha_3 \end{pmatrix} \begin{pmatrix} w_2^n \\ v_2^n \end{pmatrix},$$

$$n = 0,1,2,\ldots,$$

where the α_i are defined in (2.12). We note, that because our difference equations were obtained by reflection of the left hand side, (1.16), (1.17) can be shown to be consistent with the system

$$(2.16) \qquad \begin{pmatrix} w \\ v \end{pmatrix}_t = \begin{pmatrix} a & 0 \\ 0 & -ap \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix}_x$$

Note that the above diagonal elements are always of opposite sign. This will be very pertinent when we wish to verify condition c) of Kreiss' theorem.

We shall refer to the system of difference equations (1.16), (1.17) with boundary condition (2.15) as <u>System I</u>.

<u>Theorem</u> I. System I is a stable approximation for the Cauchy problem of Example I for $|b_1| < 1$.

<u>Proof</u>. Let $a < 0$. A similar analysis can be employed for the case $a > 0$.

As seen in Chapter I,c,d, we find all solutions to (1.16), (1.17) of the form

$$(2.17)a) \quad Z_j^n = \lambda^n \underline{\Phi}_j \; ; \quad \sum_{j=0}^{\infty} |\underline{\Phi}_j|^2 < \infty ; \quad \lambda \neq 1, \; |\lambda| \geq 1$$

by taking

$$(2.17)b) \qquad \underline{\Phi}_j = \begin{pmatrix} \beta_1 \tau_1^j \\ \beta_2 \tau_2^j \end{pmatrix}, \qquad \beta_i \text{ constants,}$$

where τ_1, τ_2 are determined respectively from

$$\text{(2.18)}\quad \begin{aligned} &\text{a)}\quad \lambda = \left(\frac{b_1^2-b_1}{2}\right)\frac{1}{\tau_1} + (1-b_1^2) + \left(\frac{b_1^2+b_1}{2}\right)\tau_1 \\ &\text{b)}\quad \lambda = \left(\frac{p^2b_1^2+pb_1}{2}\right)\frac{1}{\tau_2} + (1-p^2b_1^2) + \left(\frac{p^2b_1^2-pb_1}{2}\right)\tau_2 \end{aligned}$$

In Chapter I,c,d we showed that $\underline{\Phi}_j$ was of the form (2.17) by showing that the modes of the Lax-Wendroff scheme were separated by the unit circle for $|\lambda| \geq 1$, $\lambda \neq 1$. By Kreiss' theorem, System I is stable if it has no solution of the type (2.17) nor of the type (2.8c), that is,

$$\text{(2.19)}\quad Z_j^n = \underline{\Phi}_j \text{ where } |\underline{\Phi}_j^I| \leq \text{constant and}$$
$$|\underline{\Phi}_j^{II}| < \text{const.}|\mu|^j, \ |\mu| < 1.$$

For the case $\lambda = 1$, we can solve (2.18) directly for $\tau_{i\pm}$, $i = 1,2$. Observe that in this case, since $a < 0$,

$$|\tau_{1-}| > 1 \quad\text{and}\quad |\tau_{2-}| = 1 .$$

Referring to (2.19), we can exclude the above modes from $\underline{\Phi}_j$. Thus for the case $\lambda = 1$ the form of the $\underline{\Phi}_j$ is still (2.17). Because $a < 0$, checking (2.8c), we see that an unstable mode must be of the form (2.17) with

$$|\tau_2| < 1 .$$

We now prove stability by showing that all non-trivial solutions of system I of the form (2.17) must have $\tau_2 = 1$.

Substitution of (2.17) into the boundary condition (2.15) yields

$$(2.20)\qquad \begin{aligned} &\text{a)}\quad \beta_1 = \beta_2\tau_2 \\ &\text{b)}\quad \beta_2 = \alpha_1\beta_1\tau_1 + \alpha_2\beta_2\tau_2 + \alpha_3\beta_2\tau_2^2 \end{aligned}$$

where the τ_i are defined by (2.18) and the α_i are defined by (2.12). We are looking for a non-trivial solution for β_1, β_2. Clearly by (2.20)a $\beta_i \neq 0$ for a non-trivial solution. Using the first equation to eliminate β_1 in the second, and using the values for the α_i, we find, using that $\beta_2 \neq 0$, that

$$(2.21)\qquad p(1-p)\tau_2^2 + 2\tau_1\tau_2 + 2(p^2-1)\tau_2 - p(1+p) = 0.$$

Equating the expressions for λ from (2.18), yields upon clearing fractions

$$(2.22)\qquad \begin{aligned} &2b_1^2(1-p^2)\tau_1\tau_2 + \tau_1[p^2b_1^2+pb_1+(p^2b_1^2-pb_1)\tau_2^2-(b_1^2+b_1)\tau_1\tau_2] \\ &\qquad - (b_1^2 - b_1)\tau_2 = 0 . \end{aligned}$$

Solve for $\tau_1\tau_2$ and τ_1 from (2.21) and substitute into the above to get

$$(2.23)\quad [p(p+1)+2(1-p^2)\tau_2+p(p-1)\tau_2^2]\Big\{2b_1^2(1-p^2)\tau_2+b_1^2p^2 + b_1p+(b_1^2p^2-b_1p)\tau_2^2-(b_1^2+b_1)[\frac{p(p+1)}{2}+(1-p^2)\tau_2 + \frac{p(p-1)}{2}\tau_2^2]\Big\} - 2(b_1^2-b_1)\tau_2^2 = 0 .$$

The above expression is a polynomial of degree 4 in τ_2 with coefficients in terms of p and b_1. Let us denote the left hand side of (2.23) by

$$(2.24)\quad F(p,b,\tau_2) = A_0 + A_1\tau_2 + A_2\tau_2^2 + A_3\tau_2^3 + A_4\tau_2^4 .$$

We can determine the coefficients A_i by a straightforward inspection. We find that

$$A_0 = \left(\frac{b_1^2-b_1}{2}\right)p^2(p^2-1), \qquad A_1 = 2p^2(1-p^2)(b_1^2-b_1),$$

$$(2.25)\quad A_2 = 3(b_1^2-b_1)(p^2-1), \qquad A_3 = 2p^2(b_1^2-b_1)(1-p^2),$$

$$A_4 = \frac{p^2(p^2-1)}{2}(b_1^2-b_1) .$$

Thus (2.24) becomes

$$(2.26)\quad \begin{aligned} F(p,b_1,\tau_2) &= \left(\frac{b_1^2-b_1}{2}\right)p^2(p^2-1)(1-4\tau_2+6\tau_2^2-4\tau_2^3+\tau_2^4) \\ &= \left(\frac{b_1^2-b_1}{2}\right)p^2(p^2-1)(\tau_2-1)^4 = 0 . \end{aligned}$$

For a reasonable problem we assume that p is neither 0 nor 1. Thus, since we have assumed

$$(2.27)\quad 0 < |b_1| < 1 ,$$

we conclude from (2.26) that

$$(2.28) \qquad \tau_2 = 1 \ .$$

Q.E.D.

Remarks.

1) For the case $a > 0$, we need to alter the above argument in two places. For $\lambda = 1$, now $\tau_{1+} = 1$ and $|\tau_{2+}| > 1$. Condition c) of Kreiss' theorem dictates that $\underline{\Phi}_j$ is again of the form (2.17)b. The derivation of (2.26) can now be carried out as in our theorem. But now, an unstable mode may have $\tau_2 = 1$ if $\lambda = 1$. However, if $\tau_2 = 1$ in (2.21), we see that $\tau_1 = 1$. Since an unstable mode for this case must be of the form

$$|\tau_1| < 1$$

we have proven stability for the case $a > 0$.

2) The above proofs are valid for the case $p > 1$, since we only used the fact that p was not 0 or 1. Thus System I is stable regardless of which side is refined with respect to the other.

3) A separate verification of Kreiss' conditions shows the stability of System I when $|b_1| = 1$.

c. Boundary conditions for general mesh refinement problems.

The boundary condition employed in the previous section for System I was determined by the requirement of second-order accuracy of the Lax-Wendroff scheme at the interface of the grid patterns. The algebra of the stability proof was somewhat involved. In this section, we present different interface conditions which have the advantage that stability of the scheme is easily demonstrated. An additional advantage is that these methods are extendible to higher dimensions.

We make use of the following formula for the linear extrapolation of a sufficiently smooth function $v(x)$.

Lemma 2. Let $v_j = v(j\,\Delta x)$ then

$$v_0 = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} v_j + O(\Delta x^n) . \tag{2.29}$$

Proof. Solve for an approximation to v_0 with order of accuracy n-1 in

$$v_0 = \sum_{j=1}^{n} c_j v_j . \tag{2.30}$$

Then (2.29) follows directly by Taylor's theorem. (Or, see a treatment of divided differences e.g. Isaacson and Keller [3].)

For the case n = 3 then

$$v_0 = 3v_1 - 3v_2 + v_3 + O(\Delta x^3) \quad . \tag{2.31}$$

We now alter System I so that

$$v_0^n = w_0^n , \qquad n = 0,1,2,\ldots, \tag{2.32}$$

as in Chapter I,b. We then use (2.31) to define v_0^n and w_0^n for $n = 0,1,2,\ldots$. The resulting mesh refinement approximation scheme will be referred to as System II.

Theorem II. System II is a stable approximation to Example I when $a < 0$. If $a > 0$ then replacing (2.31) by

$$v_0^n = 3w_1^n - 3w_2^n + w_3^n , \qquad n = 0,1,2,\ldots, \tag{2.33}$$

results in a stable mesh refinement scheme for $|b_1| \leq 1$.

Remark. If the resulting approximation for v_0^{n+1} does not conform with the direction of the propagation of influence, the sufficient conditions for stability are not satisfied. That one has instabilities in such cases will be shown in numerical examples in Chapter IV.

Proof. In matrix language, our boundary condition (2.31) has the form

$$\begin{pmatrix} w_0^n \\ v_0^n \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} w_1^n \\ v_1^n \end{pmatrix} + \begin{pmatrix} 0 & -3 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} w_2^n \\ v_2^n \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_3^n \\ v_3^n \end{pmatrix} . \tag{2.34}$$

Systems I and II have the same underlying difference equations, and differ only in their interface conditions. Thus, it suffices to check solutions of the form (2.17) to (2.34) in applying Kreiss' theorem. Substitution of (2.17) into the above boundary condition gives

$$\beta_1 = \beta_2(3\tau_2 - 3\tau_2^2 + \tau_2^3) = \beta_2 \ . \tag{2.35}$$

For a non-trivial solution $\beta_2 \neq 0$ thus,

$$(\tau_2 - 1)^3 = 0 \quad \text{or} \quad \tau_2 = 1 \ . \tag{2.36}$$

Checking condition c) of Kreiss' theorem, we find for $a < 0$ that $\tau_2 = 1$ is an inadmissible mode, which proves stability. If however $a > 0$ in the above, then $\tau_2 = 1$ would be an admissible mode. Then $\lambda = \tau_2 = 1$ with $\beta_1 = \beta_2 \neq 0$ shows that the sufficient conditions for stability are not satisfied.

For the case $a > 0$, we would use (2.33) to obtain a stable mesh refinement problem. Formulating (2.33) in matrix language and checking with a solution of the form (2.17) gives us

$$\beta_1 = \beta_1(3\tau_1 - 3\tau_1^2 + \tau_1^3) = \beta_2 \quad . \tag{2.37}$$

For a non-trivial solution

$$(\tau_1 - 1)^3 = 0 \quad \text{or} \quad \tau_1 = 1.$$

This proves stability since $a > 0$. Q.E.D.

It is easy to see that boundary conditions of the form (2.29) will always give us stable approximations if the extrapolation is in terms of points which are in the domain of dependence of v_0^{n+1}. The stability of such boundary conditions has been discussed by H. O. Kreiss in [4,6].

Next we turn to a new type of interface condition, suggested by H. O. Kriess, which has the advantage of being stable regardless of the sign of a, and is also suitable for the case of higher dimensions. However, we are restricted to the case where

$$p = \frac{\Delta x_1}{\Delta x_2} = \frac{1}{M}$$

where M is an integer. Assuming a definition for v_j^n and w_j^n as in System I, we see that v_0^n coincides with w_M^n for $n = 0,1,\ldots$. Thus our boundary condition in matrix form becomes

$$(2.38) \quad \begin{pmatrix} w_0^n \\ v_0^n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1^n \\ v_1^n \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_M^n \\ v_M^n \end{pmatrix} , \quad n = 0,1,2,\ldots .$$

We shall refer to the difference approximation (to Example I) formed by using (2.38) in conjunction with the difference equations (1.16), (1.17) as System III.

<u>Theorem</u> III. System III is a stable approximation to Example I for $|b_1| \leq 1$.

Proof. Again, because of the separation of the modes, to check stability by Kreiss' theorem, we try a solution of the form (2.17). Substitution into the boundary condition (2.38) gives

$$(2.39)\qquad \begin{aligned} &a) \qquad \beta_1 = \beta_2 \tau_2 \\ &b) \qquad \beta_2 = \beta_1 \tau_1^M \end{aligned}$$

A non-trivial solution exists only if β_1 and β_2 are different from zero. Eliminating the β_i's we find

$$(2.40)\qquad \tau_2 \, \tau_1^M = 1 \, .$$

Since we look for a bounded solution, the above implies that

$$|\tau_2| = |\tau_1| = 1 \, .$$

Regardless of the sign of a, this assures stability, since by condition c) of Kreiss' theorem one of the modes must be strictly less than 1 in absolute value.

d. Matching two difference schemes along an interface.

In this section, we consider the stability of an approximation to Example I formed by using two different 3-point stable difference schemes. Let w_j^n and v_j^n denote difference approximations to $u(j\,\Delta x_1, n\,\Delta t)$ and $u(-(j-1)\Delta x_2, n\,\Delta t)$ respectively. Let S_1 and S_2 be two difference operators so that

$$(2.41)\quad \begin{aligned} &\text{a)}\quad w_j^{n+1} = S_1 w_j^n = a_{-1}w_{j-1}^n + a_0 w_j^n + a_1 w_{j+1}^n \qquad j=1,2,\ldots \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n=0,1,2,\ldots \\ &\text{b)}\quad v_j^{n+1} = S_2 v_j^n = d_{-1}v_{j-1}^n + d_0 v_j^n + d_1 v_{j+1}^n \end{aligned}$$

where a_i and d_i are constants. Assume, as in System III, that

$$p = \frac{\Delta x_1}{\Delta x_2} = \frac{1}{M}$$

where M is an integer. For Z_j^n defined by (1.15), we write our difference equations in matrix form as

$$(2.42)\text{a} \qquad Z_j^{n+1} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} Z_j^n\ , \quad j=1,2,3,\ldots;n=0,1,2,\ldots$$

At the interface of the two meshes we employ the boundary condition of System III,

$$(2.42)\text{b} \qquad Z_0^n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Z_1^n + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} Z_M^n\ , \quad n=0,1,2,\ldots\ .$$

Theorem IV. If (2.42)a is consistent with the system (2.16) and if for $0 < |\xi| \leq \pi$,

$$\text{(2.43)}\qquad \begin{aligned} &\text{a)} \quad |a_{-1}\, e^{-i\xi} + a_0 + a_1\, e^{i\xi}| < 1 \,, \\ &\text{b)} \quad |d_{-1}\, e^{-i\xi} + d_0 + d_1\, e^{i\xi}| < 1 \end{aligned}$$

then the difference approximation (2.42) is stable.

Proof. To check stability by Kreiss' theorem, we find all solutions to (2.42) of the form

$$\text{(2.44)} \qquad Z_j^n = \lambda^n \begin{pmatrix} \beta_1 \ \tau_1^j \\ \beta_2 \ \tau_2^j \end{pmatrix}, \qquad j = 0,1,2,\ldots;\ n = 0,1,2,\ldots$$

where $|\lambda| \geq 1$ and

$$\text{(2.45)}\qquad \begin{aligned} &\text{a)} \quad \lambda = a_{-1} / \tau_1 + a_0 + a_1 \tau_1 \\ &\text{b)} \quad \lambda = d_{-1} / \tau_2 + d_0 + d_1 \tau_2 \end{aligned}$$

Because (2.43) holds, Corollary 1 implies that for

$$|\lambda| \geq 1 \,, \qquad \lambda \neq 1,$$

the unit circle separates the τ_1 and τ_2 roots of (2.45). Thus, as in Theorem III, when Z_j^n is substituted into our boundary condition we have

$$|\tau_1| = |\tau_2| = 1 \,.$$

We need only check condition c) of Kreiss' theorem to complete our stability proof. Our boundary condition would imply stability as before, if we knew that the

modes were separated for the case $\lambda = 1$.

<u>Case</u> $\lambda = 1$.

We give the proof of separation of the modes for the case $a < 0$. A similar proof holds for the case $a > 0$.

Consistency means accuracy of at least first order. Taylor's theorem gives us for the difference scheme (2.41)a

$$(2.46) \qquad \begin{aligned} a_{-1} + a_0 + a_1 &= 1 \\ -a_{-1} + a_1 &= b_1 = a\,\Delta t/\Delta x_1 \end{aligned}$$

Using (2.43)a and the above we solve directly to see that in any case of interest

$$(2.47) \qquad |a_{-1}/a_1| > 1 \quad .$$

The inequality expressed in (2.47) assures us that one τ_1 root is greater than 1 in absolute value. This mode can be excluded and thus the τ_1 modes are separated. We finally note that the $\tau_2 = 1$ solution to (2.45)b (with $\lambda = 1$) is to be excluded since $a < 0$. Thus the modes are separated and, as indicated above, our boundary condition proves stability for this case. Q.E.D.

The extrapolation formula (2.29), when used so as to be consistent with the direction of propagation will also yield a stable matching of the difference schemes of Theorem IV.

e. Matrix case.

Without loss of generality, we let A in (1.1) have diagonal form as in (2.2). To form a stable mesh refinement to (1.1), we could employ a 3-point dissipative approximation in each of the M dimensions. As in the scalar case, we would reformulate the problem as a system of difference equations consistent with an initial-boundary value problem. The resulting difference equations for the mesh refinement problem are in diagonal form. The local normal mode analysis of the vector case reduces to verifying stability in each of the M dimensions in the exact manner of the scalar case. Using Kreiss' theorem, one easily verifies that Theorems I-IV hold for their matrix analogs.

CHAPTER III. EQUATIONS WITH TWO SPACE DIMENSIONS.

We consider the analogous stability problem for

$$u_t = a\, u_x + b\, u_y\,, \qquad a,b \text{ constants}, \tag{3.1}$$

in the plane $-\infty < x,y < \infty$, $t \geq 0$. We approximate the above equation for the Cauchy problem by using a difference approximation with a different mesh pattern in each half space. Let the plane $x = 0$ be the interface between the two mesh patterns. In the right half-plane we take mesh spacings in the x and y directions to be Δx_1 and Δy_1, respectively. In the left half plane we denote these respective mesh spacings by Δx_2 and Δy_2.

We restrict our attention to the case where

$$p_1 = \frac{\Delta x_1}{\Delta x_2} = \frac{1}{N} \qquad \text{and} \qquad p_2 = \frac{\Delta y_1}{\Delta y_2} = \frac{1}{M}$$

where N and M are positive integers.

Let $W^n_{j,k}$ and $V^n_{j,k}$ denote approximations to $u(j\,\Delta x_1,\ k\,\Delta y_1,\ n\,\Delta t)$ and $u(-(j-1)\Delta x_2,\ k\,\Delta y_2,\ n\,\Delta t)$ for $n = 0,1,2,\ldots$; $j = 0,1,2,\ldots$; $k = 0, \pm 1, \pm 2, \ldots$. (See figure below for an illustration of the mesh pattern.)

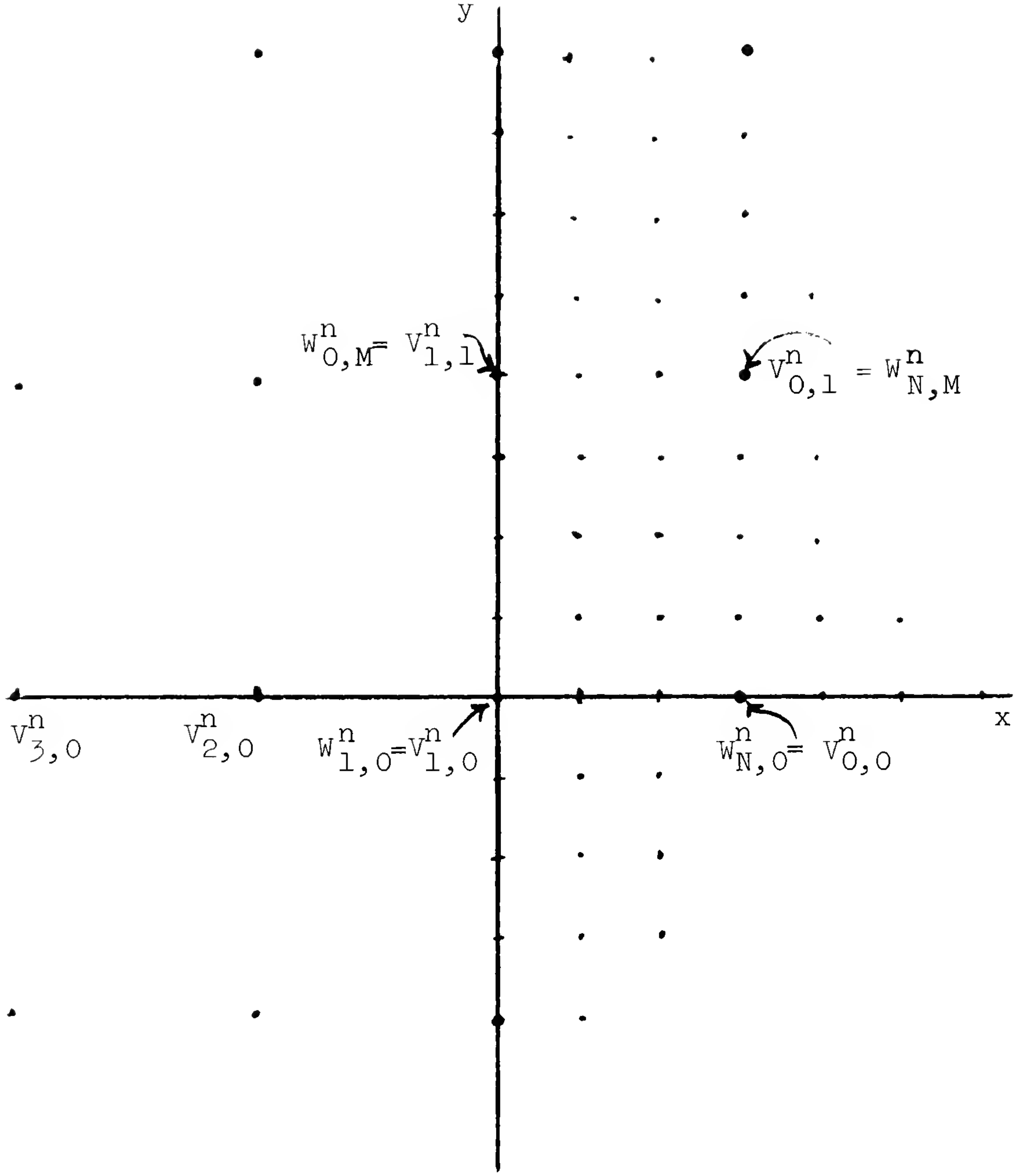

Figure 2. $N = 3$, $M = 4$.

As in the one dimensional problem, one needs to express $W_{0,k}$, $V_{0,k}$ as a linear combination of the values at neighboring points. From the definitions of our mesh functions, we have

$$(3.3) \qquad \begin{aligned} V_{0,k} &= W_{N,kM} \\ W_{0,kM} &= V_{1,k} \end{aligned} \qquad k = 0, \pm 1, \pm 2, \ldots .$$

The above exhibits the overlapping identification of some mesh points near the interface. At each time step we still have to prescribe $W^n_{0,kM+i}$ for $k = 0, \pm 1, \pm 2, \ldots$; $i = 1, 2, \ldots, M-1$. We propose to do this in terms of interpolation along the line $x = 0$. In general one could prescribe such points as a linear combination of the $V_{1,j}$ values, say,

$$(3.4) \quad W^n_{0,kM+i} = \alpha_{i,-t} V^n_{1,k-t} + \ldots + \alpha_{i,0} V^n_{1,k} + \ldots + \alpha_{s,i} V^n_{1,k+s} + O(\Delta y_2^{s+t+1})$$

for s,t natural numbers. We have used such an interpolation in our calculations for the case $s = 1$, $t = 2$. Using Taylor's theorem on (3.4) it is easy to show that one obtains for this case

$$(3.5) \qquad \begin{aligned} \alpha_{i,-2} &= -\frac{\alpha\beta(\beta+1)}{6} & \alpha_{i,-1} &= \frac{\beta(\beta+1)(\alpha+1)}{2} \\ \alpha_{i,0} &= \frac{\alpha(\alpha+1)(\beta+1)}{2} & \alpha_{i,1} &= -\frac{\alpha\beta(\alpha+1)}{6} \end{aligned}$$

where $\alpha = i/M$ and $\beta = 1-\alpha$, and $1 \le i \le M-1$.

We consider a mesh refinement problem for the Lax-Wendroff [7] 9-point difference approximation to (3.1).

$$(3.6)\quad \left.\begin{aligned} &\text{a)}\quad v_{j,k}^{n+1} = \sum_{\alpha=-1}^{1}\sum_{\beta=-1}^{1} c_{\alpha,\beta}^{(1)}\, v_{j+\alpha,k+\beta}^{n} \\ &\text{b)}\quad w_{j,k}^{n+1} = \sum_{\alpha=-1}^{1}\sum_{\beta=-1}^{1} c_{\alpha,\beta}^{(2)}\, w_{j+\alpha,k+\beta}^{n} \end{aligned}\right\}\quad \begin{aligned} n &= 0,1,2,\ldots \\ j &= 1,2,3,\ldots \\ k &= 0,\pm1,\pm2,\ldots \end{aligned}$$

where for $i = 1,2$,

$$(3.7)\quad \begin{cases} c_{0,0}^{(i)} = 1 - b_{1,i}^{2} - b_{2,i}^{2} & \\ c_{1,0}^{(i)} = \dfrac{b_{1,i}^{2} + b_{1,i}}{2} & c_{-1,0}^{(i)} = \dfrac{b_{1,i}^{2} - b_{1,i}}{2} \\ c_{0,1}^{(i)} = \dfrac{b_{2,i}^{2} + b_{2,i}}{2} & c_{0,-1}^{(i)} = \dfrac{b_{2,i}^{2} - b_{2,i}}{2} \\ \text{and } c_{j,k}^{(i)} = \dfrac{b_{1,i}\, b_{2,i}}{4}\,(j \times k) \text{ for } j=-1,1;\ k=-1,1. & \end{cases}$$

where

$$(3.8)\quad b_{1,i} = \frac{a\,\Delta t}{\Delta x_i} \quad \text{and} \quad b_{2,i} = \frac{b\,\Delta t}{\Delta y_i}\,.$$

To treat this case by the Kreiss-Osher results, we shall formulate our mesh refinement problem as a system of difference equations for an initial-boundary value problem. Because of the more complicated nature of the boundary condition in two dimensions, we found it necessary to introduce the following vector of dimension $M+1$,

$$(3.9) \qquad Z^n_{j,k} = \begin{pmatrix} V^n_{j,k} \\ W^n_{j,kM} \\ W^n_{j,kM+1} \\ \vdots \\ W^n_{j,kM+M-1} \end{pmatrix} \qquad \begin{array}{l} j = 0,1,2,\ldots \\ k = 0,\pm1,\pm2,\ldots \end{array}$$

We now consider the following system of difference equations equivalent to (3.6)a,b

$$(3.10) \qquad Z^{n+1}_{j,k} = \sum_{\alpha=-1}^{1} \sum_{\beta=-1}^{1} A_{\alpha,\beta} \, Z^n_{j+\alpha,k+\beta}$$

for $n = 0,1,2,\ldots$; $j = 1,2,3,\ldots$; $k = 0,\pm1,\pm2,\ldots$ where the matrices $A_{\alpha,\beta}$ are diagonal and given by

$$(3.11) \qquad A_{\alpha,\beta} = \begin{pmatrix} C^{(1)}_{\alpha,\beta} & & & \bigcirc \\ & C^{(2)}_{\alpha,\beta} & & \\ & & \ddots & \\ \bigcirc & & & C^{(2)}_{\alpha,\beta} \end{pmatrix}$$

The above approximation (3.9)-(3.11) is consistent with the following system of partial differential equations,

$$(3.12)\quad \begin{pmatrix} v \\ w_0 \\ w_1 \\ \vdots \\ w_{M-1} \end{pmatrix}_t = \begin{pmatrix} -ap_1 & 0 & & 0 \\ 0 & a & & \\ & & \ddots & 0 \\ 0 & & 0 & a \end{pmatrix} \begin{pmatrix} v \\ w_0 \\ w_1 \\ \vdots \\ w_{M-1} \end{pmatrix}_x$$

$$+ \begin{pmatrix} bp_2 & 0 & & 0 \\ 0 & b & & \\ & & \ddots & 0 \\ 0 & & 0 & b \end{pmatrix} \begin{pmatrix} v \\ w_0 \\ w_1 \\ \vdots \\ w_{M-1} \end{pmatrix}_y$$

(where e.g. $W^n_{j,kM+i}$ approximates $w_i(x,y,t) = u(x,y+i\Delta y_1,t)$ for $x = j\Delta x_1$, $y = kM\Delta y_1 = k\,\Delta y_2$, $t = n\,\Delta t$).

Formulating (3.3)-(3.5) as a matrix boundary condition we obtain

$$(3.13)\quad \begin{pmatrix} V^n_{0,k} \\ W^n_{0,kM} \\ \vdots \\ \\ W^n_{0,kM+M-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & & 0 \\ 1 & 0 & & 0 \\ \alpha_{1,0} & 0 & \ddots & 0 \\ \vdots & & & \\ \alpha_{M-1,0} & 0 & & 0 \end{pmatrix} Z^n_{1,k}$$

$$
+ \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \alpha_{1,1} & & \\ \vdots & & \\ \alpha_{M-1,1} & & \bigcirc \end{pmatrix} Z^n_{1,k+1} + \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & & & 0 \\ & & & & \\ & & & & \\ & & & & \bigcirc \end{pmatrix} Z^n_{N,k}
$$

$$
+ \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \alpha_{1,-1} & & \\ \vdots & & \\ \alpha_{M-1,1} & & \bigcirc \end{pmatrix} Z^n_{1,k-1} + \begin{pmatrix} 0 & \cdots & 0 \\ 0 & & 0 \\ \alpha_{1,-2} & & \\ \vdots & & \\ \alpha_{M-1,-2} & & \bigcirc \end{pmatrix} Z^n_{1,k-2}
$$

Theorem V. If the difference schemes (3.6)a,b are stable for the pure Cauchy problem then the mesh refinement scheme (3.10)-(3.13) is a stable approximation. (Note that (3.6)a,b are stable for the pure Cauchy problem if $b^2_{1,1} \le 1/8$ and $b^2_{2,1} \le 1/8$, see [7].)

Proof. To apply the Kreiss-Osher theorem, we must find all solutions to our scheme of the form

$$
(3.14) \quad Z^n_{j,k} = \lambda^n y_0^k \underline{\Phi}_j \; ; \qquad |y_0| = 1 \; ; \qquad \sum_j |\underline{\Phi}_j|^2 < \infty .
$$

As in the one dimensional case of constant coefficients, we consider a normal mode solution to (3.6)a

$$
v^n_{j,k} = \text{constant } \lambda^n \tau^j y_0^k \; ; \qquad |y_0| = 1 .
$$

Then τ satisfies the following quadratic equation,

$$
\begin{aligned}
&\left(\frac{b_{1,1}^2 + b_{1,1}}{2} + \frac{b_{1,1}\, b_{2,1}}{4}\, y_0 - \frac{b_{1,1}\, b_{2,1}}{4y_0}\right)\tau^2 \\
(3.15)\quad &+ \left(\left(1 - b_{1,1}^2 - b_{2,1}^2 - \lambda + \left(\frac{b_{2,1}^2 + b_{2,1}}{2}\right)y_0 + \left(\frac{b_{2,1}^2 - b_{2,1}}{2y_0}\right)\right)\right)\tau \\
&+ \frac{b_{1,1}^2 - b_{1,1}}{2} - \frac{b_{1,1}\, b_{2,1}}{4}\, y_0 + \frac{b_{1,1}\, b_{2,1}}{4y_0} = 0 .
\end{aligned}
$$

By the well known dissipativity of the Lax-Wendroff scheme [7], an application of the same argument as in Lemma 1 shows that for $|\lambda| \geq 1$, $\lambda \neq 1$ that the τ roots are separated by the unit circle. Furthermore, note that the case $y_0 \equiv 1$ is exactly the one dimensional case treated in Theorem I. The above proves separation of the modes for τ_1, τ_2 the respective modes of equations (3.6)a,b. This allows us to verify stability of our system by checking conditions (a-c) of (2.10) for solutions of the form

$$
(3.16)\quad Z_{j,k}^n = \lambda^n y_0^k \begin{pmatrix} \beta_0 \tau_1^j \\ \beta_1 \tau_2^j \\ \vdots \\ \beta_M \tau_2^j \end{pmatrix} ; \qquad \begin{aligned} n &= 0,1,2,\ldots \\ j &= 0,1,2,\ldots \\ k &= 0,\pm 1,\pm 2,\ldots \end{aligned}
$$

where the β_i are constants. Substitution of (3.16) into our boundary condition gives

$$
(3.17)\begin{cases}
\beta_0 = \beta_1\tau_2^N \\
\beta_1 = \beta_0\tau_1 \\
\beta_2 = \beta_0\tau_1(\alpha_{1,0}+\alpha_{1,1}y_0+\alpha_{1,-2}y_0^{-2}+\alpha_{1,-1}y_0^{-1}) \\
\vdots \\
\beta_M = \beta_0\tau_1(\alpha_{M-1,0}+\alpha_{M-1,1}y_0+\alpha_{M-1,-2}y_0^{-2}+\alpha_{M-1,-1}y_0^{-1})
\end{cases}
$$

From the above equations we observe that a non-trivial solution will exist only if β_0 and β_1 are $\neq 0$. This allows us to eliminate β_0 and β_1 in the first two equations of (3.17) and obtain

$$\tau_1\tau_2^N = 1 .$$

As in Theorem III this proves stability, since the diagonal elements of (3.12) $-ap_1$ and a are of opposite sign. Q.E.D.

Remarks.

1) The above argument can be used to prove stability for a general boundary condition of the type (3.5). For any such condition would always give a system of equations similar to (3.17) having β_0 and β_1 as the only factors on the right hand side.

2) As in Chapter I,e, we note that it is possible to extend Theorem 5 to the matrix equation

$$u_t = A\,u_x + B\,u_y \;; \quad A,B, \text{ diagonal matrices.}$$

3) The analogous two dimensional formulations of Theorems II, IV can be proven by the above method.

CHAPTER IV. COMPUTATIONAL RESULTS

Calculations were performed using the mesh refinement schemes of Theorems I-III, and V. The primary purpose was to find out the size of the inaccuracies which are introduced by the different interface conditions discussed above. Due to the hyperbolic nature of our problem, we cannot in general expect to achieve higher order accuracy in the refined regions. This is so, because the error from the unrefined region can be expected to propagate into the refined region. In fact, our calculations were performed for one period of a periodic problem. Because of this, all regions will eventually exhibit the same order of accuracy.

The mesh refinement technique might be a practical method for handling problems where large gradients in the solution are caused in a confined region. In such cases, if new inaccuracies do not arise at the interface of the mesh patterns, the mesh refinement scheme enables us to efficiently make a local adjustment to obtain higher accuracy.

a. One dimensional periodic case.

For our first series of calculations, we solved the initial value problem of Example I with $a \equiv 1$ and the solution

$$u(x,t) = \sin 4\pi(x+t) \quad . \tag{4.1}$$

We used the periodicity of our solution to restrict our calculations to the interval [0,1]. We refined the region

$$I_R \equiv \{x : \tfrac{1}{3} \le x \le \tfrac{2}{3}\} . \tag{4.2}$$

The mesh sizes were such that

$$p = \frac{\Delta x_1}{\Delta x_2} = \frac{1}{10} ; \qquad \Delta x_2 = \frac{1}{150} ; \qquad \Delta t = \frac{1}{1750} . \tag{4.3}$$

Let v_j^n and w_j^n denote the unrefined and refined schemes, respectively. We define (see Figure 3 below)

$$v_{51}^n = w_1^n ; \qquad v_{101}^n = w_{501}^n , \tag{4.4}$$

and by periodicity we always have

$$v_1^n = v_{151}^n \tag{4.5}$$

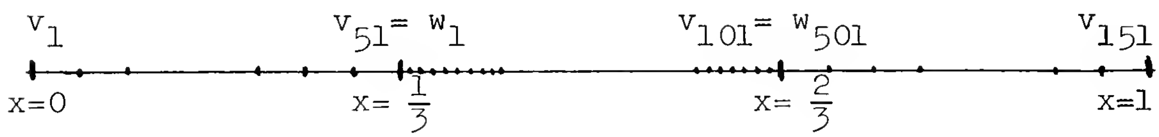

Figure 3. Refined Middle Third of Unit Interval.

Of course, we do not compute v_i^{n+1} for $i = 52,53,\ldots,100$, but use the values of w_i^{n+1}.

We describe the accuracy of the mesh refinement scheme by computing several quantities. Let $e(v)$ and $e(w)$ denote the maximum error of our scheme from the solution in the unrefined region and the interior of I_R, respectively.

The size of the error near the interfaces depends on the direction of characteristics. In our case, the error from the unrefined region advances into I_R at $x = 2/3$. To the right of $x = 1/3$ our calculations show an abruptly improved accuracy until the unrefined solution has propagated up to $x = 1/3$ from beyond $x = 2/3$. This happens after approximately 500 time steps.

To illustrate what the error in I_R would be if the characteristics always emerged from I_R, we also compute a quantity e_ℓ (for $\ell = 100$) defined by

$$(4.6) \quad e_\ell \equiv \max_{j=2,3,\ldots,\ell} |u(x,t) - w_j^n| \; ; \quad t = n\Delta t, \; x = \frac{1}{3} + (j-1)\Delta x_1 .$$

In Table 1, we list the results of several calculations for the example of (4.1) using the Lax-Wendroff scheme with refinement. The entries in rows 1-3 exhibit the stability and relatively improved accuracy in I_R of the mesh refinement scheme. The three different interface conditions give nearly identical accuracy.

The entries in rows 4 and 5 correspond to results using no refinement with $\Delta x = \Delta x_2$. For row 4 we used $\Delta t = 1/1750$. For row 5 we took our time step as $\Delta t' = 1/175$ and thus iterated only 40 times. It appears that one loses accuracy by using too small a time step (aside from having to do more work). This result is in line with a commonly observed fact that one gets better results when

Table 1

a = constant

$\Delta t = 1/1750$; $\Delta t' = 1/175$; $\Delta x_2 = 1/150$; $\Delta x_1 = 1/1500$

	t, time	e(v)	e(w)	e_{100}
1) System I	400 Δt	3.332×10^{-3}	1.591×10^{-3}	6.3×10^{-6}
2) System II	400 Δt	3.332×10^{-3}	1.656×10^{-3}	6.3×10^{-6}
3) System III	400 Δt	3.332×10^{-3}	1.629×10^{-3}	6.7×10^{-6}
4) No Refinement (Δx_2)	400 Δt	1.679×10^{-3}		
5) "	400 $\Delta t'$	8.909×10^{-4}		
6) Uneven Time (I)	40 $\Delta t'$	8.909×10^{-4}	4.493×10^{-4}	6.3×10^{-6}
7) " (III)	40 $\Delta t'$	8.909×10^{-4}	4.493×10^{-4}	6.3×10^{-6}
8) Unstable (II)	400 Δt	2.461	3.190	
9) Non-Dissipative $\Delta t^* = 4\times10^{-6}$	5000 Δt^*	4.624×10^{-4}	1.752×10^{-4}	3.3×10^{-6}

the stability factor $b_2 = a\,\Delta t/\Delta x_2$ is close to one in absolute value. In the mesh refinement problem, a small time step is needed for stability in I_R. However, using the same time step in the unrefined region will reduce accuracy there. To rectify this situation we also considered using different time steps in the different regions. In I_R we used

$$\Delta t = 1/1750 \tag{4.7}$$

as above. In the unrefined region, we used a time step

$$\Delta t' = \frac{\Delta t}{p} = \frac{1}{175} \quad . \tag{4.8}$$

The only difficulty in using this approach is at the interface. We describe how we treated the interfaces at $x = 1/3$ for Systems I and III. A similar formulation was used at $x = 2/3$.

Each cycle of computation consists of first advancing $w_1^{n/p+i}$ and $w_{501}^{n/p\ +i}$ for $i = 1,2,\ldots 10$. Then the remaining is advanced in the usual manner until all points are at the same time level. For System I we used the following modification of (1.21)

$$\text{(4.9)a}\quad w_1^{n/p+i} = \frac{b_{2,i}(b_{2,i}-p_i)}{(1+p_i)}\, v_{50}^n + \left(1 - \frac{b_{2,i}}{p_i}\right)(1+b_{2,i})v_{51}^n + \frac{b_{2,i}(b_{2,i}+1)}{p_i(p_i+1)}\, w_{11}^{n/p}$$

where

$$(4.9)\text{b} \qquad b_{2,i} = \frac{ai\,\Delta t}{\Delta x_2} \quad \text{and} \quad p_i = ip$$

for $i = 1,2,\ldots,10$.

For System III with $b_{2,i}$ as above, we used

$$(4.10) \quad w_1^{n/p+i} = \frac{(b_{2,i}^2 - b_{2,i})}{2} v_{50}^n + (1-b_{2,i}^2)v_{51}^n + \frac{(b_{2,i}^2 + b_{2,i})}{2} w_{11}^{n/p}$$

Rows 6 and 7 of Table 1 show the results obtained. Aside from the additional labor saved, the main advantage of the uneven time step approach is the improved accuracy outside the refined region.

In a remark after Theorem II, we stated that one must consider the domain of dependence of the solution when choosing an interface condition. To illustrate the instabilities which may arise when Theorem II is not observed, we calculated the scheme of row 2 as above with $a \equiv -1$. This has the effect of reversing the direction of characteristics. The interface conditions however were left unchanged. In row 8 the errors listed clearly indicate an instability.

As a further example, we calculated System II above with a different underlying difference scheme namely

$$(4.11) \quad w_j^{n+1} = \frac{-b_1}{2} w_{j-1}^n + w_j^n + \frac{b_1}{2} w_{j+1}^n .$$

Our theory does not apply to this scheme, since it is non-dissipative, does not have separation of the modes, and in fact is stable for a $\Delta t/\Delta x_1^2 \leq$ constant. Despite

this, the results of our calculations in row 9 indicate that the mesh refinement scheme is stable.

In Table 2 we present the results of computations performed for Example I with

$$(4.12)\qquad a = a(x) = \left\{\begin{array}{ll} \frac{1}{2} & 0 \le x \le \frac{1}{3} \\ \frac{2+\sin 6\pi x}{4} & x \in I_R \\ \frac{1}{2} & \frac{2}{3} \le x \le 1 \end{array}\right\}$$

The analytic solution can be found by the method of characteristics so that

$$(4.13)\qquad \begin{array}{l} u(x,t) = \Phi(\tilde{x}) \quad \text{where} \quad \Phi(x) = \sin 4\pi x \\ \text{and} \qquad t = \int_x^{\tilde{x}} \frac{d\xi}{a(\xi)} \quad . \end{array}$$

The quantities displayed in Table 2 are defined exactly in the same way as in Table 1. We also computed with several time steps for no refinement and uneven time steps. We note that in this case it was not immediately obvious whether the uneven time steps would improve accuracy in the unrefined region.

Table 2

$a = a(x)$ as in (4.1)

$\Delta t = 1/1250$; $\Delta t' = 1/125$; $\Delta x_2 = 1/150$; $\Delta x_1 = 1/1500$

	t, time	e(v)	e(w)	e_{100}
1) System I	$400\ \Delta t$	7.136×10^{-3}	1.397×10^{-3}	1.000×10^{-3}
2) System II	$400\ \Delta t$	7.134×10^{-3}	2.764×10^{-3}	1.000×10^{-3}
3) System III	$400\ \Delta t$	9.993×10^{-3}	1.401×10^{-3}	1.000×10^{-3}
4) No refinement (Δx_2)	$400\ \Delta t$	1.180×10^{-2}		
5) "	$40\ \Delta t'$	2.358×10^{-2}		
6) No refinement $\Delta t^* = 15/1250$	$27\ \Delta t^*$	2.901×10^{-2}		
7) Uneven time (I)	$40\ \Delta t' = 400\Delta t$	7.668×10^{-3}	1.652×10^{-3}	1.000×10^{-3}
8) Uneven time (III)	$40\ \Delta t' = 400\Delta t$	7.668×10^{-3}	1.611×10^{-3}	1.000×10^{-3}
9) "	$27\ \Delta t^* = 405\Delta t$	4.841×10^{-3}	1.553×10^{-3}	1.086×10^{-3}

b. Two dimensional periodic case.

Calculations were performed for the initial-value problem

$$(4.14) \qquad \begin{aligned} u_t &= -u_x + u_y \\ u(x,y,0) &= \phi(x,y) = \sin 2\pi x \cos 2\pi y . \end{aligned}$$

We used the periodicity of the solution

$$u(x,t) = \sin 2\pi(x-t) \cos 2\pi(y+t)$$

to restrict our calculations to the unit square $0 \le x,y \le 1$. The refined region was taken as (see Figure 4 below)

$$(4.15) \qquad D_R = \{(x,y): \tfrac{1}{3} \le x,y \le \tfrac{2}{3}\} .$$

Using the notation of Chapter III we took

$$(4.16) \qquad \begin{aligned} &\Delta x_2 = \Delta y_2 = \frac{1}{45} ; \quad \Delta x_1 = \Delta y_1 = \frac{1}{225} \\ &p_1 = p_2 = \frac{1}{5} \quad ; \quad \Delta t = \frac{1}{1750} . \end{aligned}$$

As in Table 1, we describe the accuracy with the analogous quantities e(v) and e(w), the maximum errors of our scheme in the unrefined region and D_R, respectively. To describe the accuracy of the refined scheme away from the interfaces, we compute the error in the interior of D_R,

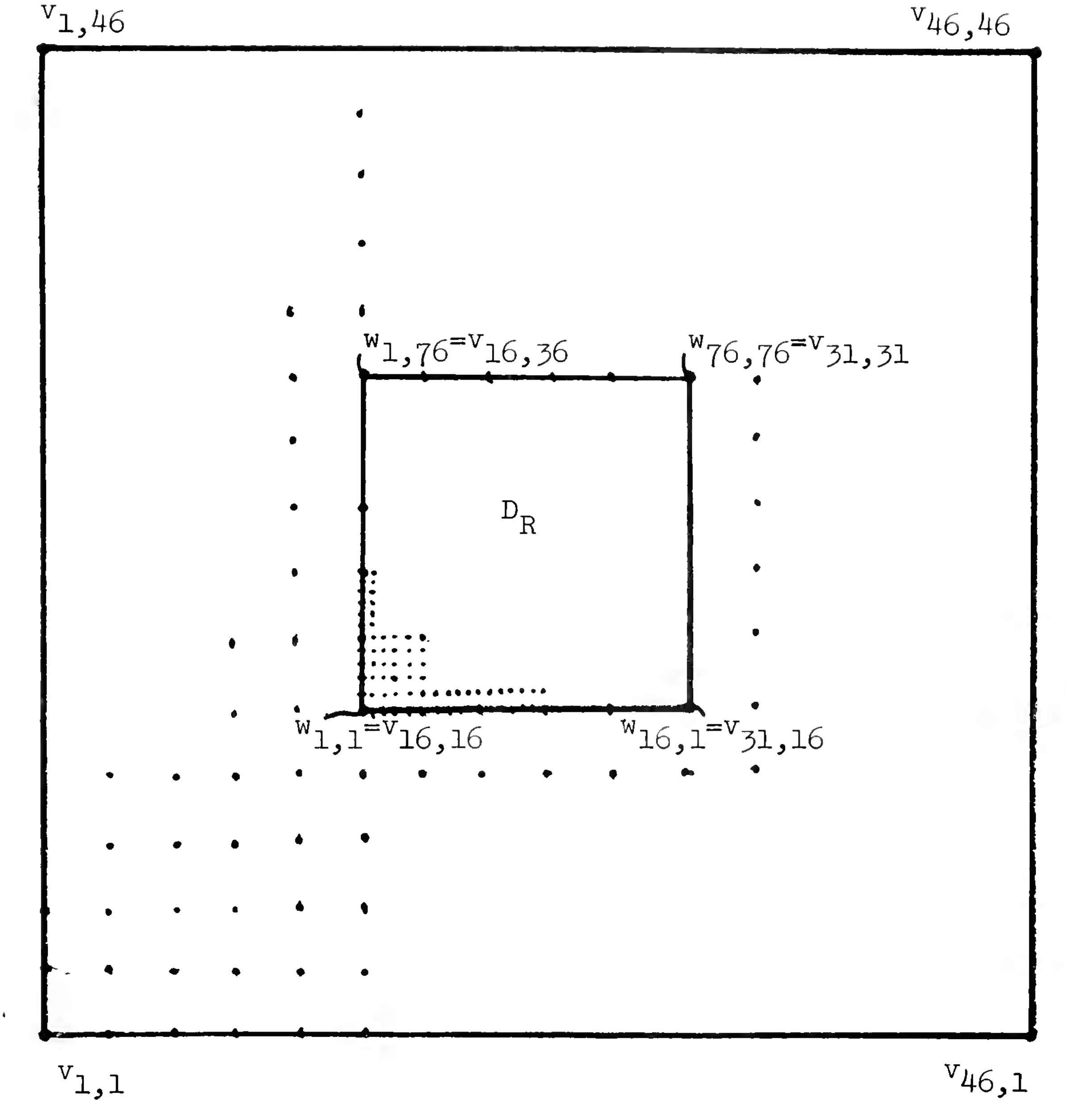

Figure 4. Unit square, $0 \leq x,y \leq 1$.

$$(4.17)\quad e_{\text{int.}} = \max_{20\le j,k\le 55} |u(x,t) - w^n_{j,k}| \; ; \quad \begin{aligned} x &= \tfrac{1}{3} + (j-1)\Delta x_1 \\ y &= \tfrac{1}{3} + (k-1)\Delta y_1 \\ t &= n\,\Delta t \end{aligned}$$

In Table 3 we list the results of our calculations using the Lax-Wendroff scheme (3.7) with our mesh refinement scheme of Theorem V. We compare the above with ordinary Lax-Wendroff with three choices of time steps. We then compute with these time steps using uneven time steps.

The procedure for computing with uneven time steps is similar to the one dimensional case. We first compute the interface by using the two dimensional analog of (4.10) at all $v^n_{j,k}$ on the interface. The only difficulty in using (3.5) is for points near the corners. At uneven time levels when interpolating for the values $w^n_{j,k}$ where $2 \le j \le 5$ or $72 \le k \le 75$, formula (3.5) calls for values of $v^n_{j,k}$ which are not available. In such cases we substituted a 3 point interpolation scheme again of the type (3.4).

Our computations show that the interface condition (3.5) does not introduce any additional errors. When uneven time steps are used, the error in the unrefined region compares with the error for ordinary no refinement. While for all entries, $e_{\text{int.}}$ remains the same. Thus, one should try to separately adjust the time steps so as to minimize the error in each region, and then match the schemes along the interface.

Table 3

$\Delta t = 1/1750$; $\Delta x_2 = \Delta y_2 = 1/45$; $\Delta x_1 = \Delta y_1 = 1/225$

	t, time	e(v)	e(w)	$e_{int.}$
1) Ordinary refinement	$100\ \Delta t$	1.182×10^{-3}	1.143×10^{-3}	4.36×10^{-5}
2) No refinement ($\Delta x_2, \Delta y_2$)				
a) $\Delta t = 1/1750$	$100\ \Delta t$	1.165×10^{-3}		
b) $\Delta t' = 5/1750$	$20\ \Delta t'$	1.104×10^{-3}		
c) $\Delta t^* = 1/175$	$10\ \Delta t^*$	8.895×10^{-4}		
3) Uneven time	$20\Delta t' = 100\Delta t$	1.104×10^{-3}	1.033×10^{-3}	4.36×10^{-5}
4) "	$10\Delta t^* = 100\Delta t$	8.895×10^{-4}	7.822×10^{-4}	4.36×10^{-5}

c. Conclusions.

In this chapter, we presented numerical results which seem to indicate the stability of the mesh refinement technique for several cases not analytically treated above.

Our main observation was the improved accuracy gained by using uneven time steps with the mesh refinement scheme. A stability proof of such a scheme involves an imposing amount of algebra. In the future we hope to prove stability only for a simple case.

We shall also try to extend these techniques to solving parabolic equations.

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